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Integrable discretizations of derivative nonlinear Schrödinger equations

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Abstract

We propose integrable discretizations of derivative nonlinear Schrödinger (DNLS) equations such as the Kaup–Newell equation, the Chen–Lee–Liu equation and the Gerdjikov–Ivanov equation by constructing Lax pairs. The discrete DNLS systems admit the reduction of complex conjugation between two dependent variables and possess bi-Hamiltonian structure. Through transformations of variables and reductions, we obtain novel integrable discretizations of the nonlinear Schrödinger (NLS), modified KdV (mKdV), mixed NLS, matrix NLS, matrix KdV, matrix mKdV, coupled NLS, coupled Hirota, coupled Sasa–Satsuma and Burgers equations. We also discuss integrable discretizations of the sine-Gordon equation, the massive Thirring model and their generalizations.

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1. Introduction

The inverse scattering method (ISM) was invented by Gardner *et al* [1] more than 30 years ago. They expressed the KdV equation as the compatibility condition of an eigenvalue problem and time evolution of the eigenfunction, and solved the KdV equation through the inverse problem of scattering. A pair of operators which defines the eigenvalue problem and the time evolution is now called the Lax pair. Zakharov and Shabat [2, 3] considered a generalization of the eigenvalue problem and solved the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} - 2uvu = O \qquad \mathrm{i}v_t - v_{xx} + 2vuv = O \tag{1.1}$$

under some conditions via the ISM. It is noteworthy that the NLS equation (1.1) is integrable for matrix-valued variables u and v [4–7]. Throughout this paper, we use the symbol italic O when dependent variables in the equation considered can take their values in matrices. Ablowitz *et al* [8] formulated the Zakharov–Shabat method in a plain manner and constructed the hierarchy of the NLS equation (1.1). Up to now, a variety of nonlinear evolution equations in continuous spacetime have been solved via the ISM based on modified versions of the Zakharov–Shabat eigenvalue problem (see, e.g., [9–17]).

In the remarkable paper [18], Ablowitz and Ladik proposed a natural discretization of the Zakharov–Shabat eigenvalue problem and obtained a discrete NLS hierarchy. Their success suggests that discrete integrable hierarchies are obtained through natural discretizations of the modified Zakharov–Shabat eigenvalue problems. It is, however, quite difficult to find such discretizations directly. In a few successful studies [19–21], the difficulty is avoided skilfully with the help of alternative approaches. Hence, wide applicability of Lax-pair formulations has not been established in the theory of discrete integrable systems.

In this paper, we discuss integrable discretizations of derivative NLS (DNLS) equations and related systems by elaborating a new formulation of Lax pairs. Among the DNLS equations, the following three representatives are well known [22–24], i.e. the Kaup–Newell equation [15],

$$\mathrm{i}q_t^{\mathrm{K}} + q_{xx}^{\mathrm{K}} - \mathrm{i}(q^{\mathrm{K}}r^{\mathrm{K}}q^{\mathrm{K}})_x = O \qquad \mathrm{i}r_t^{\mathrm{K}} - r_{xx}^{\mathrm{K}} - \mathrm{i}(r^{\mathrm{K}}q^{\mathrm{K}}r^{\mathrm{K}})_x = O \tag{1.2}$$

the Chen–Lee–Liu equation [25],

$$\mathrm{i}q_t^{\mathrm{C}} + q_{xx}^{\mathrm{C}} - \mathrm{i}q_x^{\mathrm{C}}r^{\mathrm{C}}q^{\mathrm{C}} = O \qquad \mathrm{i}r_t^{\mathrm{C}} - r_{xx}^{\mathrm{C}} - \mathrm{i}r^{\mathrm{C}}q^{\mathrm{C}}r_x^{\mathrm{C}} = O \tag{1.3}$$

and the Gerdjikov–Ivanov equation [26],

$$\mathrm{i}q_t^{\mathrm{G}} + q_{xx}^{\mathrm{G}} + \mathrm{i}q^{\mathrm{G}}r_x^{\mathrm{G}}q^{\mathrm{G}} + \tfrac{1}{2}q^{\mathrm{G}}r^{\mathrm{G}}q^{\mathrm{G}}r^{\mathrm{G}}q^{\mathrm{G}} = O \qquad \mathrm{i}r_t^{\mathrm{G}} - r_{xx}^{\mathrm{G}} + \mathrm{i}r^{\mathrm{G}}q_x^{\mathrm{G}}r^{\mathrm{G}} - \tfrac{1}{2}r^{\mathrm{G}}q^{\mathrm{G}}r^{\mathrm{G}}q^{\mathrm{G}}r^{\mathrm{G}} = O. \tag{1.4}$$

It has been shown that (1.2)–(1.4) are integrable for matrix-valued variables [27–29] (see [30, 31] for complete lists of integrable coupled DNLS equations). The DNLS equations (1.2)–(1.4) correspond to special cases ($\delta = -1/2, -1/4, 0$, respectively) of the generalized DNLS equation [32]

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + \mathrm{i}(4\delta+1)q^2r_x + 4\mathrm{i}\delta qq_xr + (\delta+1/2)(4\delta+1)q^3r^2 = 0 \\ &\mathrm{i}r_t - r_{xx} + \mathrm{i}(4\delta+1)r^2q_x + 4\mathrm{i}\delta rr_xq - (\delta+1/2)(4\delta+1)r^3q^2 = 0. \end{aligned} \tag{1.5}$$

The zero symbol 0 indicates that the dependent variables are restricted to scalars. We generate (1.5) from the Gerdjikov–Ivanov equation (1.4) via the transformation

$$q = q^{\mathrm{G}}\exp\left(-2\mathrm{i}\delta\int^x q^{\mathrm{G}}r^{\mathrm{G}}\,\mathrm{d}x'\right) \qquad r = r^{\mathrm{G}}\exp\left(2\mathrm{i}\delta\int^x q^{\mathrm{G}}r^{\mathrm{G}}\,\mathrm{d}x'\right). \tag{1.6}$$

A comprehensive description of physical applications of the DNLS equations as well as their exact solutions can be found in [23, 24]. An integrable discretization of the Chen–Lee–Liu equation (1.3) was proposed by Date, Jimbo and Miwa [33]. However, their scheme does not admit the reduction of complex conjugation between q_n^{C} and r_n^{C}. Therefore it is of little practical use. In other studies on discrete DNLS equations [34–36], equations of motion are not explicitly given. Practically, little is known about natural integrable discretizations of the DNLS equations.

To solve this problem, we consider discrete analogues of relations between the DNLS hierarchies and a generalization of the NLS hierarchy [15, 22, 37, 38]. We extend the Lax-pair formulation of Ablowitz and Ladik to obtain a generalization of the discrete NLS (Ablowitz–Ladik) system. We find that integrable discretizations of the DNLS equations (1.2)–(1.4) together with their higher symmetries are embedded in the generalized Ablowitz–Ladik system. With appropriate gauge transformations, we obtain standard forms of eigenvalue problems for the discrete DNLS systems. Using a discrete version of the transformation (1.6), we obtain a discrete generalized DNLS system. All the discrete DNLS systems with one exception admit the reduction of complex conjugation. Thus they inherit the crucial property of the continuous DNLS systems. Through simple transformations and reductions for the discrete DNLS systems, we obtain integrable discretizations of various systems, e.g. the mixed NLS, matrix NLS, matrix KdV, matrix modified KdV (matrix mKdV), coupled Sasa–Satsuma and Burgers equations. In this way, we can derive integrable discretizations of surprisingly many systems from one generalization of the Ablowitz–Ladik formulation. This exemplifies that some approaches based on a Lax-pair formulation are very fruitful in the study of discrete integrable systems.

The sine-Gordon equation and its generalization are integrable via the ISM based on the Zakharov–Shabat eigenvalue problem [39, 40]. In contrast, Ablowitz and Ladik were unsuccessful in obtaining a local discretization of the sine-Gordon equation (cf (2.10) in [18]). This does not indicate that their discretization of the Zakharov–Shabat eigenvalue problem is useless in studying discrete sine-Gordon equations. It is just because time evolution of the eigenfunction assumed in [18] is too general and inappropriate. We prove that discrete sine-Gordon equations and their generalizations are associated with the Ablowitz–Ladik eigenvalue problem (see appendices A and B). The massive Thirring model and its variant types are known to have the same eigenvalue problems as the DNLS equations have [13–15, 27, 29, 37]. In particular, the massive Thirring model and the Chen–Lee–Liu equation (1.3) have the eigenvalue problem in common. This indicates that the massive Thirring-type models are related to some generalization of the sine-Gordon equation. Combining these items of information, we derive integrable discretizations of the massive Thirring-type models from the eigenvalue problems for the discrete DNLS systems.

This paper consists of the following. In section 2, we show that the DNLS equations are embedded in a generalization of the NLS equation, and propose a generalization of the Ablowitz–Ladik system. In section 3, we reveal that space discretizations (in short, semi-discretizations) of the DNLS systems are embedded in the generalized Ablowitz–Ladik system. In section 4, we obtain a variety of integrable lattice systems related to the semi-discrete DNLS systems. In section 5, we investigate semi-discretizations of the massive Thirring-type models. Section 6 is devoted to concluding remarks.

In the present version of this paper, we do not discuss time discretizations of the semi-discrete DNLS systems. Some of the results on this problem can be found in old versions of this paper (arXiv:nlin.SI/0105053 ver. 1 or 2).

2. Preliminaries

In this section, we first show that the DNLS equations (1.2)–(1.4) are embedded in a generalization of the NLS equation (1.1). Next, we extend the Ablowitz–Ladik formulation to obtain a semi-discretization of the generalized NLS system.

2.1. Embeddings of the DNLS equations into a generalized NLS equation

Let us begin with a brief description of Lax pairs. We consider a pair of linear equations

$$\Psi_x = U\Psi \qquad \Psi_t = V\Psi \tag{2.1}$$

for a column vector Ψ. The subscripts x and t denote the partial differentiations by x and t, respectively. U and V are square matrices which depend on a parameter. The compatibility condition of (2.1) is given by

$$U_t - V_x + UV - VU = O \tag{2.2}$$

which we call the zero-curvature condition. If we choose the matrices U, V appropriately, (2.2) gives a system independent of the parameter. In such cases, the pair of matrices U, V and the parameter are called the Lax pair and the spectral parameter, respectively.

We introduce the following form of the Lax pair:

$$U = \mathrm{i}\lambda \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \begin{bmatrix} w & u \\ v & s \end{bmatrix} \tag{2.3a}$$

$$V = \mathrm{i}\lambda^2 \begin{bmatrix} -2I_1 & \\ & 2I_2 \end{bmatrix} + \lambda \begin{bmatrix} & 2u \\ 2v & \end{bmatrix} + \mathrm{i} \begin{bmatrix} c & u_x - wu + us \\ -v_x - vw + sv & d \end{bmatrix}. \tag{2.3b}$$

Here λ is the spectral parameter. U and V are divided into four blocks as $(l_1 + l_2) \times (l_1 + l_2)$ matrices. I_1 and I_2 are, respectively, the $l_1 \times l_1$ and $l_2 \times l_2$ unit matrices. w, s, u, v are matrix-valued variables. c and d are arbitrary functions at this stage. Substituting (2.3) into the zero-curvature condition (2.2), we obtain the following system (see (26) in [41] for the case of scalar variables):

$$\begin{aligned}
&\mathrm{i}w_t + c_x - wc + cw + (uv)_x + uvw - wuv = O \\
&\mathrm{i}s_t + d_x - sd + ds - (vu)_x - vus + svu = O \\
&\mathrm{i}u_t + u_{xx} - ud + cu - w_x u - 2wu_x + 2u_x s + us_x + w^2 u - 2wus + us^2 = O \\
&\mathrm{i}v_t - v_{xx} - vc + dv + s_x v + 2sv_x - 2v_x w - vw_x - s^2 v + 2svw - vw^2 = O.
\end{aligned} \tag{2.4}$$

We call (2.4) the generalized NLS equation. If we set $w = O, s = O, c = -uv, d = vu$, (2.4) is reduced to the matrix NLS equation (1.1).

We already know Lax pairs for the matrix DNLS equations (1.2)–(1.4) [29]. With the help of gauge transformations, we can transform the Lax pairs into the form (2.3) (see, e.g., [42]). Thus the DNLS equations are embedded in the generalized NLS equation (2.4). We omit details and show the main results in the following. For simplicity of the embedding formulae, we change the scalings of variables in (1.2)–(1.4).

(a) If q^{K} and r^{K} satisfy the Kaup–Newell equation,

$$\mathrm{i}q^{\mathrm{K}}_t + q^{\mathrm{K}}_{xx} + 2(q^{\mathrm{K}} r^{\mathrm{K}} q^{\mathrm{K}})_x = O \qquad \mathrm{i}r^{\mathrm{K}}_t - r^{\mathrm{K}}_{xx} + 2(r^{\mathrm{K}} q^{\mathrm{K}} r^{\mathrm{K}})_x = O \tag{2.5}$$

w, s, u, v defined by

$$w = -q^{\mathrm{K}} r^{\mathrm{K}} \qquad s = r^{\mathrm{K}} q^{\mathrm{K}} \qquad u = q^{\mathrm{K}} \qquad v = r^{\mathrm{K}}_x - r^{\mathrm{K}} q^{\mathrm{K}} r^{\mathrm{K}}$$

satisfy the generalized NLS equation (2.4) with

$$c = -q^{\mathrm{K}}_x r^{\mathrm{K}} - 2q^{\mathrm{K}} r^{\mathrm{K}} q^{\mathrm{K}} r^{\mathrm{K}} \qquad d = r^{\mathrm{K}} q^{\mathrm{K}}_x + 2r^{\mathrm{K}} q^{\mathrm{K}} r^{\mathrm{K}} q^{\mathrm{K}}.$$

(b) If q^{C} and r^{C} satisfy the Chen–Lee–Liu equation,

$$\mathrm{i}q^{\mathrm{C}}_t + q^{\mathrm{C}}_{xx} + 2q^{\mathrm{C}}_x r^{\mathrm{C}} q^{\mathrm{C}} = O \qquad \mathrm{i}r^{\mathrm{C}}_t - r^{\mathrm{C}}_{xx} + 2r^{\mathrm{C}} q^{\mathrm{C}} r^{\mathrm{C}}_x = O \tag{2.6}$$

w, s, u, v defined by

$$w = O \qquad s = r^{\mathrm{C}} q^{\mathrm{C}} \qquad u = q^{\mathrm{C}} \qquad v = r^{\mathrm{C}}_x$$

satisfy the generalized NLS equation (2.4) with

$$c = -q^{\mathrm{C}} r^{\mathrm{C}}_x \qquad d = r^{\mathrm{C}} q^{\mathrm{C}}_x + r^{\mathrm{C}} q^{\mathrm{C}} r^{\mathrm{C}} q^{\mathrm{C}}.$$

(c) If q^{G} and r^{G} satisfy the Gerdjikov–Ivanov equation,

$$\begin{aligned}
&\mathrm{i}q^{\mathrm{G}}_t + q^{\mathrm{G}}_{xx} - 2q^{\mathrm{G}} r^{\mathrm{G}}_x q^{\mathrm{G}} - 2q^{\mathrm{G}} r^{\mathrm{G}} q^{\mathrm{G}} r^{\mathrm{G}} q^{\mathrm{G}} = O \\
&\mathrm{i}r^{\mathrm{G}}_t - r^{\mathrm{G}}_{xx} - 2r^{\mathrm{G}} q^{\mathrm{G}}_x r^{\mathrm{G}} + 2r^{\mathrm{G}} q^{\mathrm{G}} r^{\mathrm{G}} q^{\mathrm{G}} r^{\mathrm{G}} = O
\end{aligned} \tag{2.7}$$

u and v defined by

$$u = q^{\mathrm{G}} \qquad v = r^{\mathrm{G}}_x + r^{\mathrm{G}} q^{\mathrm{G}} r^{\mathrm{G}}$$

satisfy the matrix NLS equation (1.1).

2.2. Generalization of the Ablowitz–Ladik system

We consider a semi-discrete version of the linear problem (2.1):

$$\Psi_{n+1} = L_n \Psi_n \qquad \Psi_{n,t} = M_n \Psi_n. \tag{2.8}$$

The compatibility condition of (2.8) is given by

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O \tag{2.9}$$

which is a semi-discrete version of the zero-curvature condition (2.2).

We generalize the Lax pair proposed by Ablowitz and Ladik [18] and consider the following form:

$$L_n = \begin{bmatrix} z w_n & u_n \\ v_n & \frac{1}{z} s_n \end{bmatrix} \qquad M_n = \begin{bmatrix} z^2 a I_1 + c_n & z a w_n^{-1} u_n + \frac{b}{z} u_{n-1} s_{n-1}^{-1} \\ z a v_{n-1} w_{n-1}^{-1} + \frac{b}{z} s_n^{-1} v_n & d_n + \frac{b}{z^2} I_2 \end{bmatrix}. \tag{2.10}$$

Here z is the spectral parameter and a, b are constants. L_n and M_n are divided into four blocks as $(l_1 + l_2) \times (l_1 + l_2)$ matrices. I_1 and I_2 are unit matrices. w_n, s_n, u_n, v_n are matrix-valued variables. c_n and d_n are arbitrary functions. Substitution of the Lax pair (2.10) into the zero-curvature condition (2.9) yields the following system:

$$\begin{aligned} &w_{n,t} + w_n c_n - c_{n+1} w_n + a u_n v_{n-1} w_{n-1}^{-1} - a w_{n+1}^{-1} u_{n+1} v_n = O \\ &s_{n,t} + s_n d_n - d_{n+1} s_n + b v_n u_{n-1} s_{n-1}^{-1} - b s_{n+1}^{-1} v_{n+1} u_n = O \\ &u_{n,t} + u_n d_n - c_{n+1} u_n + b w_n u_{n-1} s_{n-1}^{-1} - a w_{n+1}^{-1} u_{n+1} s_n = O \\ &v_{n,t} + v_n c_n - d_{n+1} v_n + a s_n v_{n-1} w_{n-1}^{-1} - b s_{n+1}^{-1} v_{n+1} w_n = O. \end{aligned} \tag{2.11}$$

We call (2.11) the generalized Ablowitz–Ladik system. When just two of w_n, s_n, u_n, v_n are functionally independent, we may determine c_n, d_n from the consistency of (2.11). If we set $w_n = I_1, s_n = I_2, c_n = -a(I_1 + u_n v_{n-1}), d_n = -b(I_2 + v_n u_{n-1})$, (2.11) is reduced to the matrix Ablowitz–Ladik system [43–47]:

$$\begin{aligned} &u_{n,t} - a u_{n+1} + b u_{n-1} + (a-b) u_n + a u_{n+1} v_n u_n - b u_n v_n u_{n-1} = O \\ &v_{n,t} - b v_{n+1} + a v_{n-1} + (b-a) v_n + b v_{n+1} u_n v_n - a v_n u_n v_{n-1} = O. \end{aligned} \tag{2.12}$$

The system (2.12) with $b = a^*$ admits the reduction of complex conjugation $v_n = \sigma u_n^*$ and not the reduction of Hermitian conjugation $v_n = \sigma u_n^\dagger$. Throughout this paper we use σ to denote an arbitrary (but usually nonzero) real constant. We can construct an infinite set of conservation laws for (2.11) by a recursive method [44, 45]. The first five conserved densities are

$$\operatorname{tr}\log w_n \qquad \operatorname{tr}\log s_n \qquad \operatorname{tr}\log\left(w_n - u_n s_n^{-1} v_n\right) \qquad \operatorname{tr}\left(u_n v_{n-1} w_{n-1}^{-1} w_n^{-1}\right) \qquad \operatorname{tr}\left(v_n u_{n-1} s_{n-1}^{-1} s_n^{-1}\right).$$

The generalized Ablowitz–Ladik system (2.11) gives a semi-discretization of the generalized NLS equation (2.4) together with its higher symmetry. To see this, we suppose the following scalings for (2.11):

$$\begin{aligned} &w_n(t) = I_1 + \Delta w(x,t) \qquad s_n(t) = I_2 + \Delta s(x,t) \qquad u_n(t) = \Delta u(x,t) \qquad v_n(t) = \Delta v(x,t) \\ &a = -b = \frac{\mathrm{i}}{\Delta^2} \qquad c_n(t) = -\frac{\mathrm{i}}{\Delta^2} I_1 + \mathrm{i}c(x,t) \qquad d_n(t) = \frac{\mathrm{i}}{\Delta^2} I_2 + \mathrm{i}d(x,t). \end{aligned}$$

Here Δ denotes the lattice spacing and $x = n\Delta$. In the continuum limit $\Delta \to 0$, (2.11) is precisely reduced to (2.4). A correspondence between the spectral parameters is given by $z = \exp(-\mathrm{i}\lambda\Delta)$. In the next section, we show that semi-discrete DNLS systems are embedded in (2.11) through lattice analogues of the formulae (a)–(c) in section 2.1. We also mention that a system of semi-discrete coupled NLS equations is obtained as a reduction of (2.11) [45].

3. Semi-discrete DNLS systems

In this section, we obtain semi-discrete DNLS systems by considering lattice analogues of (a)–(c) for the generalized Ablowitz–Ladik system (2.11). Through a discrete analogue of the transformation (1.6), we obtain a semi-discrete generalized DNLS system.

3.1. Semi-discrete Kaup–Newell system

We find that a lattice analogue of (a) is given as follows.

(A) If q_n^{K} and r_n^{K} satisfy the system

$$q_{n,t}^{\mathrm{K}} - a\left(I_1 - q_{n+1}^{\mathrm{K}} r_{n+1}^{\mathrm{K}}\right)^{-1} q_{n+1}^{\mathrm{K}} + a\left(I_1 - q_n^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} q_n^{\mathrm{K}} - b\left(I_1 + q_n^{\mathrm{K}} r_{n+1}^{\mathrm{K}}\right)^{-1} q_n^{\mathrm{K}} + b\left(I_1 + q_{n-1}^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} q_{n-1}^{\mathrm{K}} = O \tag{3.1a}$$

$$r_{n,t}^{\mathrm{K}} - b\left(I_2 + r_{n+1}^{\mathrm{K}} q_n^{\mathrm{K}}\right)^{-1} r_{n+1}^{\mathrm{K}} + b\left(I_2 + r_n^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1} r_n^{\mathrm{K}} - a\left(I_2 - r_n^{\mathrm{K}} q_n^{\mathrm{K}}\right)^{-1} r_n^{\mathrm{K}} + a\left(I_2 - r_{n-1}^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1} r_{n-1}^{\mathrm{K}} = O \tag{3.1b}$$

w_n, s_n, u_n, v_n defined by

$$w_n = I_1 - q_n^{\mathrm{K}} r_n^{\mathrm{K}} \qquad s_n = I_2 + r_{n+1}^{\mathrm{K}} q_n^{\mathrm{K}} \qquad u_n = q_n^{\mathrm{K}} \qquad v_n = r_{n+1}^{\mathrm{K}} - r_n^{\mathrm{K}} - r_{n+1}^{\mathrm{K}} q_n^{\mathrm{K}} r_n^{\mathrm{K}} \tag{3.2}$$

satisfy the generalized Ablowitz–Ladik system (2.11) with

$$\begin{aligned} c_n &= -bI_1 - a\left(I_1 - q_n^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} + b\left(I_1 + q_{n-1}^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} \\ d_n &= -aI_2 + a\left(I_2 - r_n^{\mathrm{K}} q_n^{\mathrm{K}}\right)^{-1} - b\left(I_2 + r_n^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1}. \end{aligned} \tag{3.3}$$

The system (3.1) with $a = -b = \mathrm{i}$ gives an integrable semi-discretization of the Kaup–Newell equation (2.5), while (3.1) with $a = b = -1$ gives an integrable lattice version of a higher symmetry of (2.5) (cf (3.39) in [27]).

To consider the reduction of complex conjugation or Hermitian conjugation, we suppose that q_n^{K} and r_n^{K} are defined on the fractional lattice $n \in \mathbb{Z}/2$. If $b = a^*$, the semi-discrete Kaup–Newell system (3.1) admits either the reduction $r_n^{\mathrm{K}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{K}*}$ or the reduction $r_n^{\mathrm{K}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{K}\dagger}$. This property is crucial for various applications of (3.1).

Through the gauge transformation

$$\Phi_n = \begin{bmatrix} f(z)I_1 & O \\ -\frac{1}{z} r_n^{\mathrm{K}} & I_2 \end{bmatrix} \Psi_n \tag{3.4}$$

where $f(z)$ is an arbitrary function of z, the linear problem (2.8) and the Lax pair (2.10) with (3.2) and (3.3) are changed into

$$\Phi_{n+1} = L_n^{\mathrm{K}} \Phi_n \qquad \Phi_{n,t} = M_n^{\mathrm{K}} \Phi_n$$

and

$$L_n^{\mathrm{K}} = \begin{bmatrix} zI_1 - \left(z - \frac{1}{z}\right) q_n^{\mathrm{K}} r_n^{\mathrm{K}} & f(z) q_n^{\mathrm{K}} \\ \frac{1}{f(z)}\left(-1 + \frac{1}{z^2}\right) r_n^{\mathrm{K}} & \frac{1}{z} I_2 \end{bmatrix} = \begin{bmatrix} zI_1 & zf(z) q_n^{\mathrm{K}} \\ O & I_2 \end{bmatrix} \begin{bmatrix} I_1 & O \\ \frac{1}{f(z)}\left(-1 + \frac{1}{z^2}\right) r_n^{\mathrm{K}} & \frac{1}{z} I_2 \end{bmatrix} \tag{3.5a}$$

$$M_n^{\mathrm{K}} = \left[\begin{array}{c|c} \left[a(z^2-1)+b\left(-1+\frac{1}{z^2}\right)\right] I_1 & f(z)\left[az\left(I_1 - q_n^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} q_n^{\mathrm{K}} \right. \\ +b\left(1-\frac{1}{z^2}\right)\left(I_1 + q_{n-1}^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} & \left. +\frac{b}{z}\left(I_1 + q_{n-1}^{\mathrm{K}} r_n^{\mathrm{K}}\right)^{-1} q_{n-1}^{\mathrm{K}}\right] \\ \hline \frac{1}{f(z)}\left[a\left(-z+\frac{1}{z}\right)\left(I_2 - r_{n-1}^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1} r_{n-1}^{\mathrm{K}} \right. & b\left(-1+\frac{1}{z^2}\right)\left(I_2 + r_n^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1} \\ \left. +b\left(-\frac{1}{z}+\frac{1}{z^3}\right)\left(I_2 + r_n^{\mathrm{K}} q_{n-1}^{\mathrm{K}}\right)^{-1} r_n^{\mathrm{K}}\right] & \end{array}\right]. \tag{3.5b}$$

We can check that substitution of the Lax pair (3.5) into the zero-curvature condition (2.9) yields the semi-discrete Kaup–Newell system (3.1). If we set

$$f(z) = \left(-1+\frac{1}{z^2}\right)^{\frac{1}{2}} \qquad z = \exp(-\mathrm{i}\zeta^2\Delta) \qquad q_n^{\mathrm{K}} = \left(-\frac{\mathrm{i}\Delta}{2}\right)^{\frac{1}{2}} q_n \qquad r_n^{\mathrm{K}} = \left(-\frac{\mathrm{i}\Delta}{2}\right)^{\frac{1}{2}} r_n$$

the L_n-matrix (3.5*a*) has the following asymptotic form:

$$L_n^{\mathrm{K}} = \begin{bmatrix} I_1 & \\ & I_2 \end{bmatrix} + \Delta \begin{bmatrix} -\mathrm{i}\zeta^2 I_1 & \zeta q_n \\ \zeta r_n & \mathrm{i}\zeta^2 I_2 \end{bmatrix} + O(\Delta^2).$$

Consequently, in the continuum limit $\Delta \to 0$, we recover the eigenvalue problem proposed by Kaup and Newell [15].

For the time being, we assume without loss of generality that q_n^{K} and r_n^{K} are square matrices. If necessary, we append dummy variables to q_n^{K} and r_n^{K} which are finally equalized to zero. Then a set of Hamiltonian and Poisson brackets for the semi-discrete Kaup–Newell system (3.1) is given by

$$H = \sum_n \left[-a \log\det\left(I - q_n^{\mathrm{K}} r_n^{\mathrm{K}}\right) + b \log\det\left(I + q_n^{\mathrm{K}} r_{n+1}^{\mathrm{K}}\right)\right] \tag{3.6a}$$

and

$$\left\{q_n^{\mathrm{K}} \underset{,}{\otimes} q_m^{\mathrm{K}}\right\} = \left\{r_n^{\mathrm{K}} \underset{,}{\otimes} r_m^{\mathrm{K}}\right\} = O \qquad \left\{q_n^{\mathrm{K}} \underset{,}{\otimes} r_m^{\mathrm{K}}\right\} = (\delta_{n+1,m} - \delta_{n,m})\Pi. \tag{3.6b}$$

Here Π denotes the permutation matrix: $\Pi_{kl}^{ij} = \delta_{i,l}\delta_{j,k}$. It is easily proved that $q_{n,t}^{\mathrm{K}} = \left\{q_n^{\mathrm{K}}, H\right\}$ and $r_{n,t}^{\mathrm{K}} = \left\{r_n^{\mathrm{K}}, H\right\}$ coincide with (3.1*a*) and (3.1*b*), respectively.

Let us prove that in the periodic boundary case $\left(q_{n+M}^{\mathrm{K}} = q_n^{\mathrm{K}}, r_{n+M}^{\mathrm{K}} = r_n^{\mathrm{K}}\right)$ (3.1) has an involutive set of conserved quantities. We introduce new variables p_n^{K} by $q_n^{\mathrm{K}} = p_{n+1}^{\mathrm{K}} - p_n^{\mathrm{K}}$ and $p_{n+M}^{\mathrm{K}} = p_n^{\mathrm{K}}$, which satisfy ultra-local Poisson brackets:

$$\left\{p_n^{\mathrm{K}} \underset{,}{\otimes} p_m^{\mathrm{K}}\right\} = \left\{r_n^{\mathrm{K}} \underset{,}{\otimes} r_m^{\mathrm{K}}\right\} = O \qquad \left\{p_n^{\mathrm{K}} \underset{,}{\otimes} r_m^{\mathrm{K}}\right\} = \delta_{n,m}\Pi.$$

For convenience, we set $f(z) = 1/z$. Applying a gauge transformation to the L_n-matrix (3.5*a*), we obtain a new ultra-local L_n-matrix:

$$\begin{aligned} \mathcal{L}_n^{\mathrm{K}}(z) &= \begin{bmatrix} -I & p_{n+1}^{\mathrm{K}} \\ O & I \end{bmatrix} L_n^{\mathrm{K}} \begin{bmatrix} -I & p_n^{\mathrm{K}} \\ O & I \end{bmatrix} \\ &= \begin{bmatrix} zI & \\ & \frac{1}{z}I \end{bmatrix} + \left(z - \frac{1}{z}\right)\begin{bmatrix} p_n^{\mathrm{K}} r_n^{\mathrm{K}} & -p_n^{\mathrm{K}} - p_n^{\mathrm{K}} r_n^{\mathrm{K}} p_n^{\mathrm{K}} \\ r_n^{\mathrm{K}} & -r_n^{\mathrm{K}} p_n^{\mathrm{K}} \end{bmatrix}. \end{aligned}$$

The matrix $\mathcal{L}_n^{\mathrm{K}}(z)$ satisfies the fundamental r-matrix relation:

$$\left\{\mathcal{L}_n^{\mathrm{K}}(\lambda) \underset{,}{\otimes} \mathcal{L}_m^{\mathrm{K}}(\mu)\right\} = \delta_{n,m}\left[\mathcal{L}_n^{\mathrm{K}}(\lambda) \otimes \mathcal{L}_m^{\mathrm{K}}(\mu), r(\lambda,\mu)\right]. \tag{3.7}$$

Here $r(\lambda, \mu)$ in the case of scalar variables is given by

$$r(\lambda,\mu) = -\frac{(\lambda^2-1)(\mu^2-1)}{\lambda^2-\mu^2}\begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix}. \tag{3.8}$$

$r(\lambda, \mu)$ for matrix-valued variables is given by replacing the ones in the matrix (3.8) with the permutation matrix Π. This is similar to the r-matrix given in [7].

The fundamental r-matrix relation (3.7) results in commutativity of the traces of the monodromy matrix $\mathcal{L}_M^{\mathrm{K}}\mathcal{L}_{M-1}^{\mathrm{K}}\cdots\mathcal{L}_1^{\mathrm{K}}$ for different values of the spectral parameter. This generates a set of M conserved quantities in involution. Owing to the relation $\mathrm{tr}\left(\mathcal{L}_M^{\mathrm{K}}\mathcal{L}_{M-1}^{\mathrm{K}}\cdots\mathcal{L}_1^{\mathrm{K}}\right) = \mathrm{tr}\left(L_M^{\mathrm{K}}L_{M-1}^{\mathrm{K}}\cdots L_1^{\mathrm{K}}\right)$ we can write the conserved quantities in terms of the original variables q_n^{K} and r_n^{K}. The set of conserved quantities includes the Hamiltonian (3.6*a*) itself.

3.2. Semi-discrete Chen–Lee–Liu system

We find that a lattice analogue of (b) is given as follows.

(B) If q_n^{C} and r_n^{C} satisfy the system

$$\begin{aligned} &q_{n,t}^{\mathrm{C}} - a\left(q_{n+1}^{\mathrm{C}} - q_n^{\mathrm{C}}\right)\left(I_2 + r_n^{\mathrm{C}}q_n^{\mathrm{C}}\right) - b\left(q_n^{\mathrm{C}} - q_{n-1}^{\mathrm{C}}\right)\left(I_2 + r_n^{\mathrm{C}}q_{n-1}^{\mathrm{C}}\right)^{-1} = O \\ &r_{n,t}^{\mathrm{C}} - b\left(I_2 + r_{n+1}^{\mathrm{C}}q_n^{\mathrm{C}}\right)^{-1}\left(r_{n+1}^{\mathrm{C}} - r_n^{\mathrm{C}}\right) - a\left(I_2 + r_n^{\mathrm{C}}q_n^{\mathrm{C}}\right)\left(r_n^{\mathrm{C}} - r_{n-1}^{\mathrm{C}}\right) = O \end{aligned} \tag{3.9}$$

w_n, s_n, u_n, v_n defined by

$$w_n = I_1 \qquad s_n = I_2 + r_{n+1}^{\mathrm{C}}q_n^{\mathrm{C}} \qquad u_n = q_n^{\mathrm{C}} \qquad v_n = r_{n+1}^{\mathrm{C}} - r_n^{\mathrm{C}} \tag{3.10}$$

satisfy the generalized Ablowitz–Ladik system (2.11) with

$$c_n = -aI_1 - aq_n^{\mathrm{C}}\left(r_n^{\mathrm{C}} - r_{n-1}^{\mathrm{C}}\right) \qquad d_n = -b\left(I_2 + r_n^{\mathrm{C}}q_{n-1}^{\mathrm{C}}\right)^{-1} + ar_n^{\mathrm{C}}q_n^{\mathrm{C}}. \tag{3.11}$$

The system (3.9) with $a = -b = \mathrm{i}$ gives an integrable semi-discretization of the Chen–Lee–Liu equation (2.6), while (3.9) with $a = b = -1$ gives an integrable lattice version of a higher symmetry of (2.6) (cf (3.37) in [27]).

Unfortunately, the semi-discrete Chen–Lee–Liu system (3.9) admits neither the reduction $r_n^{\mathrm{C}} = \mathrm{i}\sigma q_{n-k}^{\mathrm{C}*}$ nor the reduction $r_n^{\mathrm{C}} = \mathrm{i}\sigma q_{n-k}^{\mathrm{C}\dagger}$. This is an intrinsic drawback for practical use. We obtain an alternative semi-discretization of the Chen–Lee–Liu system, which admits the reduction of complex conjugation, in section 3.5.

Through the gauge transformation (3.4), replacing r_n^{K} by r_n^{C}, the linear problem (2.8) and the Lax pair (2.10) with (3.10) and (3.11) are changed into

$$\Phi_{n+1} = L_n^{\mathrm{C}}\Phi_n \qquad \Phi_{n,t} = M_n^{\mathrm{C}}\Phi_n$$

and

$$L_n^{\mathrm{C}} = \begin{bmatrix} zI_1 + \frac{1}{z}q_n^{\mathrm{C}}r_n^{\mathrm{C}} & f(z)q_n^{\mathrm{C}} \\ \frac{1}{f(z)}\left(-1 + \frac{1}{z^2}\right)r_n^{\mathrm{C}} & \frac{1}{z}I_2 \end{bmatrix} \tag{3.12a}$$

$$M_n^{\mathrm{C}} = \left[\begin{array}{c|c} \begin{array}{l}\left[a(z^2-2) + \frac{b}{z^2}\right]I_1 + a\left(I_1 + q_n^{\mathrm{C}}r_{n-1}^{\mathrm{C}}\right) \\ -\frac{b}{z^2}\left(I_1 + q_{n-1}^{\mathrm{C}}r_n^{\mathrm{C}}\right)^{-1}\end{array} & \begin{array}{l} f(z)\Big[azq_n^{\mathrm{C}} \\ \quad + \frac{b}{z}\left(I_1 + q_{n-1}^{\mathrm{C}}r_n^{\mathrm{C}}\right)^{-1}q_{n-1}^{\mathrm{C}}\Big]\end{array} \\ \hline \begin{array}{l}\frac{1}{f(z)}\Big[a\left(-z + \frac{1}{z}\right)r_{n-1}^{\mathrm{C}} \\ \quad + b\left(-\frac{1}{z} + \frac{1}{z^3}\right)\left(I_2 + r_n^{\mathrm{C}}q_{n-1}^{\mathrm{C}}\right)^{-1}r_n^{\mathrm{C}}\Big]\end{array} & b\left(-1 + \frac{1}{z^2}\right)\left(I_2 + r_n^{\mathrm{C}}q_{n-1}^{\mathrm{C}}\right)^{-1} \end{array}\right]. \tag{3.12b}$$

We can prove that substitution of the Lax pair (3.12) into the zero-curvature condition (2.9) yields the semi-discrete Chen–Lee–Liu system (3.9). With appropriate scalings, we reproduce the eigenvalue problem for the Chen–Lee–Liu system in the continuum limit.

3.3. Semi-discrete Gerdjikov–Ivanov system

We find that a lattice analogue of (c) is given as follows.

(C) If q_n^{G} and r_n^{G} satisfy the system

$$\begin{aligned} q_{n,t}^{\mathrm{G}} - aq_{n+1}^{\mathrm{G}} + bq_{n-1}^{\mathrm{G}} + (a-b)q_n^{\mathrm{G}} + aq_{n+1}^{\mathrm{G}}\left(r_{n+1}^{\mathrm{G}} - r_n^{\mathrm{G}}\right)q_n^{\mathrm{G}} - bq_n^{\mathrm{G}}\left(r_{n+1}^{\mathrm{G}} - r_n^{\mathrm{G}}\right)q_{n-1}^{\mathrm{G}} \\ + aq_{n+1}^{\mathrm{G}}r_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}q_n^{\mathrm{G}} - bq_n^{\mathrm{G}}r_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}} = O \end{aligned} \tag{3.13a}$$

$$\begin{aligned} r_{n,t}^{\mathrm{G}} - br_{n+1}^{\mathrm{G}} + ar_{n-1}^{\mathrm{G}} + (b-a)r_n^{\mathrm{G}} - br_{n+1}^{\mathrm{G}}\left(q_n^{\mathrm{G}} - q_{n-1}^{\mathrm{G}}\right)r_n^{\mathrm{G}} + ar_n^{\mathrm{G}}\left(q_n^{\mathrm{G}} - q_{n-1}^{\mathrm{G}}\right)r_{n-1}^{\mathrm{G}} \\ + br_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}}r_n^{\mathrm{G}} - ar_n^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}}r_{n-1}^{\mathrm{G}} = O \end{aligned} \tag{3.13b}$$

u_n and v_n defined by

$$u_n = q_n^{\mathrm{G}} \qquad v_n = r_{n+1}^{\mathrm{G}} - r_n^{\mathrm{G}} + r_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}} \tag{3.14}$$

satisfy the matrix Ablowitz–Ladik system (2.12).

If $q_n^{\mathrm{G}} = I$, the relation (3.14) between r_n^{G} and v_n is nothing but a discrete Miura transformation [48]. The system (3.13) with $a = -b = \mathrm{i}$ gives an integrable semi-discretization of the Gerdjikov–Ivanov equation (2.7), while (3.13) with $a = b = -1$ gives an integrable lattice version of a higher symmetry of (2.7) (cf (3.45) in [27]). The semi-discrete Gerdjikov–Ivanov system (3.13) with $b = a^*$ admits the reduction $r_n^{\mathrm{G}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{G}*}$ and not the reduction $r_n^{\mathrm{G}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{G}\dagger}$.

Through the gauge transformation (3.4) with replacing r_n^{K} by r_n^{G}, the linear problem (2.8) and the Lax pair for (2.12) are changed into

$$\Phi_{n+1} = L_n^{\mathrm{G}}\Phi_n \qquad \Phi_{n,t} = M_n^{\mathrm{G}}\Phi_n$$

and

$$L_n^{\mathrm{G}} = \begin{bmatrix} zI_1 + \frac{1}{z}q_n^{\mathrm{G}}r_n^{\mathrm{G}} & f(z)q_n^{\mathrm{G}} \\ \frac{1}{f(z)}\left(-1+\frac{1}{z^2}\right)\left(r_n^{\mathrm{G}} - r_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}\right) & \frac{1}{z}I_2 - \frac{1}{z}r_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}} \end{bmatrix} \tag{3.15a}$$

$$M_n^{\mathrm{G}} = \left[\begin{array}{c|c} \begin{array}{l} a(z^2-1)I_1 + aq_n^{\mathrm{G}}r_{n-1}^{\mathrm{G}} \\ +\frac{b}{z^2}q_{n-1}^{\mathrm{G}}r_n^{\mathrm{G}} - aq_n^{\mathrm{G}}r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}}r_{n-1}^{\mathrm{G}} \end{array} & f(z)\left(azq_n^{\mathrm{G}} + \frac{b}{z}q_{n-1}^{\mathrm{G}}\right) \\ \hline \begin{array}{l} \frac{1}{f(z)}\left(-1+\frac{1}{z^2}\right)\left(I_2 - r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}}\right) \\ \times\left(azr_{n-1}^{\mathrm{G}} + \frac{b}{z}r_n^{\mathrm{G}}\right) \end{array} & \begin{array}{l} b\left(-1+\frac{1}{z^2}\right)\left(I_2 - r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}}\right) - ar_n^{\mathrm{G}}q_n^{\mathrm{G}} \\ -br_{n+1}^{\mathrm{G}}q_{n-1}^{\mathrm{G}} - br_{n+1}^{\mathrm{G}}q_n^{\mathrm{G}}r_n^{\mathrm{G}}q_{n-1}^{\mathrm{G}} \end{array} \end{array}\right]. \tag{3.15b}$$

Putting the Lax pair (3.15) into the zero-curvature condition (2.9), we exactly obtain the semi-discrete Gerdjikov–Ivanov system (3.13). The L_n-matrix (3.15a) depends on q_n^{G}, r_n^{G} and r_{n+1}^{G}, thus it is not ultra-local. It is an open question whether the system (3.13) possesses a Lax pair with an ultra-local L_n-matrix. In the continuum limit of space, we regenerate the eigenvalue problem for the Gerdjikov–Ivanov system.

By a generalization of the transformation (3.14)

$$u_n = q_n \qquad v_n = \lambda r_{n+1} - r_n + \lambda r_{n+1}q_nr_n$$

for (2.12), we obtain a generalization of (3.13):

$$\begin{aligned} q_{n,t} - aq_{n+1} + bq_{n-1} + (a-b)q_n + aq_{n+1}(\lambda r_{n+1} - r_n)q_n - bq_n(\lambda r_{n+1} - r_n)q_{n-1} \\ + a\lambda q_{n+1}r_{n+1}q_nr_nq_n - b\lambda q_nr_{n+1}q_nr_nq_{n-1} = O \end{aligned} \tag{3.16a}$$

$$\begin{aligned} r_{n,t} - br_{n+1} + ar_{n-1} + (b-a)r_n - br_{n+1}(q_n - \lambda q_{n-1})r_n + ar_n(q_n - \lambda q_{n-1})r_{n-1} \\ + b\lambda r_{n+1}q_nr_nq_{n-1}r_n - a\lambda r_nq_nr_nq_{n-1}r_{n-1} = O. \end{aligned} \tag{3.16b}$$

This system is considered as a one-parameter deformation of the matrix Ablowitz–Ladik system (2.12). The change of variables $q_n = q_n^{\mathrm{G}} \lambda^n \mathrm{e}^{[(\lambda-1)a+(1-\lambda^{-1})b]t}$, $r_n = r_n^{\mathrm{G}} \lambda^{-n} \mathrm{e}^{-[(\lambda-1)a+(1-\lambda^{-1})b]t}$ and scalings of a, b cast (3.16) into (3.13). Transformations of this kind are very useful for applications of the discrete DNLS systems and we utilize them in sections 4.1 and 4.2.

3.4. Semi-discrete generalized DNLS system

In sections 3.4 and 3.5, we assume that dependent variables are scalar. As a discrete analogue of (1.6), we consider the following transformation for the semi-discrete Gerdjikov–Ivanov system (3.13):

$$q_n = q_n^{\mathrm{G}} \prod_{j=-\infty}^{n} \left(\frac{1 - q_{j-1}^{\mathrm{G}} r_j^{\mathrm{G}}}{1 + q_j^{\mathrm{G}} r_j^{\mathrm{G}}} \right)^{-2\gamma} \qquad r_n = r_n^{\mathrm{G}} \prod_{j=-\infty}^{n} \left(\frac{1 - q_{j-1}^{\mathrm{G}} r_j^{\mathrm{G}}}{1 + q_{j-1}^{\mathrm{G}} r_{j-1}^{\mathrm{G}}} \right)^{2\gamma}. \tag{3.17}$$

Here γ is a real parameter. It should be noted that both $\log\left(1 - q_{n-1}^{\mathrm{G}} r_n^{\mathrm{G}}\right)$ and $\log\left(1 + q_n^{\mathrm{G}} r_n^{\mathrm{G}}\right)$ are conserved densities for (3.13). The form of the transformation is so chosen that the reduction $r_n^{\mathrm{G}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{G}*}$ results in $r_n = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{*}$.

The inverse of the transformation (3.17) is written as

$$q_n^{\mathrm{G}} = q_n \prod_{j=-\infty}^{n} \frac{g_\gamma(q_j r_j)}{g_\gamma(-q_{j-1} r_j)} \qquad r_n^{\mathrm{G}} = r_n \prod_{j=-\infty}^{n} \frac{g_\gamma(-q_{j-1} r_j)}{g_\gamma(q_{j-1} r_{j-1})}. \tag{3.18}$$

Here $g_\gamma(y)$ is defined by the functional equation

$$[1 + y g_\gamma(y)]^{-2\gamma} = g_\gamma(y).$$

By the successive approximation, $g_\gamma(y)$ for a general value of γ is expressed as the power series:

$$g_\gamma(y) = 1 - 2\gamma y + \gamma(6\gamma + 1) y^2 + \cdots. \tag{3.19}$$

Substituting (3.18) into the semi-discrete Gerdjikov–Ivanov system (3.13), we obtain an integrable lattice version of the generalized DNLS system:

$$\begin{aligned} q_{n,t} &- a q_{n+1} g_\gamma(q_{n+1} r_{n+1}) \left[\frac{1}{g_\gamma(-q_n r_{n+1})} - q_n r_{n+1} \right] [1 + (2\gamma + 1) q_n r_n g_\gamma(q_n r_n)] \\ &+ b q_{n-1} g_\gamma(-q_{n-1} r_n) \left[\frac{1}{g_\gamma(q_n r_n)} + q_n r_n \right] [1 - (2\gamma + 1) q_n r_{n+1} g_\gamma(-q_n r_{n+1})] \\ &+ (a - b) q_n - 2\gamma b q_n r_{n+1} q_n g_\gamma(-q_n r_{n+1}) - 2\gamma a q_n r_n q_n g_\gamma(q_n r_n) = 0 \end{aligned} \tag{3.20a}$$

$$\begin{aligned} r_{n,t} &- b r_{n+1} g_\gamma(-q_n r_{n+1}) \left[\frac{1}{g_\gamma(q_n r_n)} + q_n r_n \right] [1 - (2\gamma + 1) q_{n-1} r_n g_\gamma(-q_{n-1} r_n)] \\ &+ a r_{n-1} g_\gamma(q_{n-1} r_{n-1}) \left[\frac{1}{g_\gamma(-q_{n-1} r_n)} - q_{n-1} r_n \right] [1 + (2\gamma + 1) q_n r_n g_\gamma(q_n r_n)] \\ &+ (b - a) r_n + 2\gamma a r_n q_n r_n g_\gamma(q_n r_n) + 2\gamma b r_n q_{n-1} r_n g_\gamma(-q_{n-1} r_n) = 0. \end{aligned} \tag{3.20b}$$

System (3.20) with $a = -b = \mathrm{i}$ gives an integrable semi-discretization of (1.5). As was expected from the construction of (3.17), (3.20) with $b = a^*$ admits the reduction $r_n = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{*}$. Although (3.20) is not an explicit expression for general γ, we can obtain approximate expressions by using the power series (3.19).

For some choices of γ, $g_\gamma(y)$ is written explicitly, e.g.

$$g_{-1}(y) = \left(\frac{2}{1+\sqrt{1-4y}}\right)^2 \qquad g_{-\frac{1}{2}}(y) = \frac{1}{1-y} \qquad g_{-\frac{1}{4}}(y) = \sqrt{1+\frac{1}{4}y^2}+\frac{1}{2}y$$

$$g_0(y) = 1 \qquad g_{\frac{1}{2}}(y) = \frac{2}{1+\sqrt{1+4y}}.$$

When $\gamma = -1/2$, (3.20) coincides with the semi-discrete Kaup–Newell system (3.1) for scalar variables.

3.5. Alternative semi-discrete Chen–Lee–Liu system

Setting $\gamma = -1/4$ in (3.20), we obtain an integrable semi-discretization of the Chen–Lee–Liu system which is different from (3.9). We write the equation for q_n:

$$\begin{aligned} q_{n,t} - a &\left\{1+(q_nr_n)^2+q_nr_n\sqrt{1+(q_nr_n)^2}\right\}\left\{\left[\sqrt{1+(q_{n+1}r_{n+1})^2}+q_{n+1}r_{n+1}\right]\left[\sqrt{1+(q_nr_{n+1})^2}\right.\right.\\ &\qquad \left.\left. - q_nr_{n+1}\right]q_{n+1}-q_n\right\} + b\left\{1+(q_nr_{n+1})^2-q_nr_{n+1}\sqrt{1+(q_nr_{n+1})^2}\right\}\\ &\qquad \times\left\{\left[\sqrt{1+(q_{n-1}r_n)^2}-q_{n-1}r_n\right]\left[\sqrt{1+(q_nr_n)^2}+q_nr_n\right]q_{n-1}-q_n\right\} = 0. \end{aligned} \tag{3.21}$$

Here we have changed the scaling of q_nr_n to eliminate fractions. The alternative semi-discrete Chen–Lee–Liu system ((3.21) and the equation for r_n) with $b = a^*$ admits the reduction $r_n = \mathrm{i}\sigma q^*_{n-\frac{1}{2}}$. In this respect, the alternative system is superior to the semi-discrete Chen–Lee–Liu system (3.9).

System (3.9) for scalar variables is related to the semi-discrete Gerdjikov–Ivanov system (3.13) via the transformation

$$q_n^{\mathrm{C}} = q_n^{\mathrm{G}}\prod_{j=-\infty}^{n}\left(1-q_{j-1}^{\mathrm{G}}r_j^{\mathrm{G}}\right) \qquad r_n^{\mathrm{C}} = r_n^{\mathrm{G}}\prod_{j=-\infty}^{n}\frac{1}{1-q_{j-1}^{\mathrm{G}}r_j^{\mathrm{G}}}.$$

These relations among the semi-discrete DNLS systems (3.1), (3.9) and (3.13) can be generalized to the case of matrix-valued variables (cf [29]).

4. Systems related to the semi-discrete DNLS

In this section, we obtain integrable semi-discretizations of various systems through transformations and reductions for the semi-discrete DNLS systems.

4.1. Semi-discrete mixed NLS systems

The so-called mixed NLS equations are obtained as a mixture of an integrable DNLS equation and the NLS equation [16, 22–24, 30, 49, 50]. To give an actual example, we add $+cq^2r$ and $-cr^2q$ to the left-hand sides of (1.5), respectively. Since the mixed NLS equations are connected with the original DNLS equations through transformations of variables, they are also integrable. By a semi-discrete version of such transformations

$$q_n \to q_n\,\mathrm{e}^{\mathrm{i}(\theta n+\varphi t)} \qquad r_n \to r_n\,\mathrm{e}^{-\mathrm{i}(\theta n+\varphi t)}$$

for the semi-discrete DNLS systems, we straightforwardly obtain integrable semi-discretizations of the mixed NLS systems.

4.2. Semi-discrete NLS equations and semi-discrete mKdV equations

For special choices of the parameters θ, φ, the semi-discrete mixed NLS systems are reduced to novel lattice versions of the NLS equation and the mKdV equation.

Substituting $q_n^{\mathrm{K}} = (-1)^n \mathrm{e}^{2(b-a)t} q_n$, $r_n^{\mathrm{K}} = (-1)^n \mathrm{e}^{2(a-b)t} r_n$ into the semi-discrete Kaup–Newell system (3.1), we obtain

$$q_{n,t} + a(I_1 - q_{n+1}r_{n+1})^{-1}q_{n+1} + a(I_1 - q_n r_n)^{-1}q_n - b(I_1 - q_n r_{n+1})^{-1}q_n \\ - b(I_1 - q_{n-1}r_n)^{-1}q_{n-1} + 2(b-a)q_n = O \tag{4.1a}$$

$$r_{n,t} + b(I_2 - r_{n+1}q_n)^{-1}r_{n+1} + b(I_2 - r_n q_{n-1})^{-1}r_n - a(I_2 - r_n q_n)^{-1}r_n \\ - a(I_2 - r_{n-1}q_{n-1})^{-1}r_{n-1} + 2(a-b)r_n = O. \tag{4.1b}$$

Putting $q_n^{\mathrm{C}} = (-1)^n \mathrm{e}^{2(b-a)t} q_n$, $r_n^{\mathrm{C}} = (-1)^n \mathrm{e}^{2(a-b)t} r_n$ into the semi-discrete Chen–Lee–Liu system (3.9), we obtain

$$\begin{aligned} &q_{n,t} + a(q_{n+1} + q_n)(I_2 + r_n q_n) - b(q_n + q_{n-1})(I_2 - r_n q_{n-1})^{-1} + 2(b-a)q_n = O \\ &r_{n,t} + b(I_2 - r_{n+1}q_n)^{-1}(r_{n+1} + r_n) - a(I_2 + r_n q_n)(r_n + r_{n-1}) + 2(a-b)r_n = O. \end{aligned} \tag{4.2}$$

Substitution of $q_n^{\mathrm{G}} = (-1)^n \mathrm{e}^{2(b-a)t} q_n$, $r_n^{\mathrm{G}} = (-1)^n \mathrm{e}^{2(a-b)t} r_n$ into the semi-discrete Gerdjikov–Ivanov system (3.13) yields

$$q_{n,t} + aq_{n+1} - bq_{n-1} + (b-a)q_n + aq_{n+1}(r_{n+1} + r_n)q_n - bq_n(r_{n+1} + r_n)q_{n-1} \\ + aq_{n+1}r_{n+1}q_n r_n q_n - bq_n r_{n+1}q_n r_n q_{n-1} = O \tag{4.3a}$$

$$r_{n,t} + br_{n+1} - ar_{n-1} + (a-b)r_n + br_{n+1}(q_n + q_{n-1})r_n - ar_n(q_n + q_{n-1})r_{n-1} \\ + br_{n+1}q_n r_n q_{n-1} r_n - ar_n q_n r_n q_{n-1} r_{n-1} = O. \tag{4.3b}$$

Systems (4.1)–(4.3) give integrable semi-discretizations of the matrix NLS equation for $a = -b = -\mathrm{i}$ and the matrix mKdV equation for $a = b = 1$, respectively.

In the continuous theory, the matrix NLS equation (1.1) and the matrix mKdV equation (see (4.24) in [4] or (2.12) in [7]) admit either of the reductions $v = \sigma u^*$ and $v = \sigma u^\dagger$. The system (4.3) with $b = a^*$ admits the reduction $r_n = \sigma q^*_{n-\frac{1}{2}}$ and not the reduction $r_n = \sigma q^\dagger_{n-\frac{1}{2}}$. This is similar to the property of the matrix Ablowitz–Ladik system (2.12) explained in section 2.2. The system (4.1) with $b = a^*$ admits the reduction $r_n = \sigma q^\dagger_{n-\frac{1}{2}}$ as well as the reduction $r_n = \sigma q^*_{n-\frac{1}{2}}$. In this respect, the differential-difference scheme (4.1) faithfully inherits the property of the continuous hierarchy. This enables us to obtain integrable semi-discretizations of reductions of the matrix NLS hierarchy, e.g. the coupled NLS equations [5, 9], the coupled Hirota equations [6, 30, 51, 52] and the coupled Sasa–Satsuma equations (cf section 4.3).

4.3. Semi-discrete Sasa–Satsuma equations

If dependent variables are real matrices, the Hermitian conjugation coincides with the transposition. The matrix mKdV equation in this case takes the following form [6, 7, 53]:

$$u_t + u_{xxx} - 3\sigma\left(u_x{}^t u u + u\,{}^t u u_x\right) = O. \tag{4.4}$$

Let us consider the vector reduction: $u = (u_1, \ldots, u_{2m}) \in \mathbb{R}^{2m}$ [4, 10]. Defining a set of complex variables by $\psi_k = u_{2k-1} + \mathrm{i}u_{2k} (k = 1, \ldots, m)$, we obtain the coupled Sasa–Satsuma equations [10, 30, 54, 55] (cf (4.26) in [4]),

$$\psi_t + \psi_{xxx} - 3\sigma \|\psi\|^2 \psi_x - \tfrac{3}{2}\sigma (\|\psi\|^2)_x \psi = \mathbf{0}. \tag{4.5}$$

Here $\psi = (\psi_1, \ldots, \psi_m) \in \mathbb{C}^m$.

The system (4.1) with $a = b = 1$ and $r_n = \sigma\, {}^t q_{n-\frac{1}{2}}$ gives an integrable semi-discretization of the real matrix mKdV equation (4.4). Following the same procedure as in the continuous case, we obtain an integrable lattice version of the coupled Sasa–Satsuma equations (4.5).

4.4. Semi-discrete KdV equations

If we consider the reduction $a = b = 1, q_n = I + p_n, r_n = I$ for (4.1), we obtain a matrix generalization of a semi-discrete KdV equation [48, 56],

$$p_{n,t} = p_{n+1}^{-1} - p_{n-1}^{-1}.$$

Through the reduction $a = b = 1, q_n = -I + y_n, r_n = I$ for (4.3), we obtain a matrix version of a semi-discrete KdV equation [48],

$$y_{n,t} + y_{n+1} y_n^2 - y_n^2 y_{n-1} = O.$$

4.5. Semi-discrete Burgers systems

The matrix Burgers equation,

$$q_t^{\mathrm{c}} - q_{xx}^{\mathrm{c}} + 2q_x^{\mathrm{c}} q^{\mathrm{c}} = O \tag{4.6}$$

is obtained as the reduction $r^{\mathrm{c}} = -I$ for the Chen–Lee–Liu equation (2.6) (after scalings). The Hopf–Cole transformation $q^{\mathrm{c}} = -F_x F^{-1}$ casts the linear diffusion equation $F_t - F_{xx} = O$ into (4.6).

In a similar way, we obtain integrable semi-discretizations of the Burgers equation together with its higher symmetry. Equating r_n^{c} with $-I$ in the semi-discrete Chen–Lee–Liu system (3.9), we obtain

$$q_{n,t}^{\mathrm{c}} - a\left(q_{n+1}^{\mathrm{c}} - q_n^{\mathrm{c}}\right)\left(I - q_n^{\mathrm{c}}\right) - b\left(q_n^{\mathrm{c}} - q_{n-1}^{\mathrm{c}}\right)\left(I - q_{n-1}^{\mathrm{c}}\right)^{-1} = O. \tag{4.7}$$

The substitution $I - q_n^{\mathrm{c}} = F_{n+1} F_n^{-1}$ relates (4.7) with the linear equation

$$F_{n,t} - aF_{n+1} + bF_{n-1} + (a - b)F_n = O.$$

Setting $r_n = -1$ in the alternative semi-discrete Chen–Lee–Liu system (3.21), we obtain

$$\begin{aligned} q_{n,t} - \sqrt{1+q_n^2}\, \Big\{ & a\left[\sqrt{1+q_{n+1}^2} - q_{n+1}\right] q_{n+1} - a\left[\sqrt{1+q_n^2} - q_n\right] q_n \\ & + b\left[\sqrt{1+q_n^2} + q_n\right] q_n - b\left[\sqrt{1+q_{n-1}^2} + q_{n-1}\right] q_{n-1} \Big\} = 0. \end{aligned} \tag{4.8}$$

The substitution $\left[\sqrt{1+q_n^2} - q_n\right]^2 = f_{n+1} f_n^{-1}$ connects (4.8) with the linear equation

$$f_{n,t} - af_{n+1} + bf_{n-1} + (a - b)f_n = 0.$$

The systems (4.7) and (4.8) give two lattice versions of the Burgers equation for $a = -b = 1$ and its higher symmetry for $a = b = 1$, respectively.

5. Semi-discrete massive Thirring-type models

In section 3, we obtained the Lax pairs for the semi-discrete DNLS systems. If we can replace the M_n-matrices with appropriate ones, we obtain integrable semi-discretizations of the massive Thirring-type models. For this purpose, the Lax pair introduced in appendix A provides useful information.

5.1. Semi-discrete Kaup–Newell type

For the semi-discrete Kaup–Newell system in section 3.1, we replace the M_n-matrix (3.5*b*) with the following one:

$$M_n^{\mathrm{K}} = \frac{1}{z-\frac{1}{z}} \begin{bmatrix} \frac{1}{z} I_1 - 2\left(z-\frac{1}{z}\right)\phi_n^{\mathrm{K}}\chi_n^{\mathrm{K}} & 2\mathrm{i} f(z)\phi_n^{\mathrm{K}} \\ \frac{2\mathrm{i}}{f(z)}\left(-1+\frac{1}{z^2}\right)\chi_n^{\mathrm{K}} & -zI_2 + 2\left(z-\frac{1}{z}\right)\chi_n^{\mathrm{K}}\phi_n^{\mathrm{K}} \end{bmatrix}. \tag{5.1}$$

Here ϕ_n^{K} and χ_n^{K} are new variables which are, respectively, $l_1 \times l_2$ and $l_2 \times l_1$ matrices. Substituting the Lax pair (3.5*a*) and (5.1) into the zero-curvature condition (2.9), we obtain

$$\begin{aligned} &q_{n,t}^{\mathrm{K}} + \mathrm{i}\left(\phi_n^{\mathrm{K}} + \phi_{n+1}^{\mathrm{K}}\right) + 2\left(q_n^{\mathrm{K}}\chi_{n+1}^{\mathrm{K}}\phi_n^{\mathrm{K}} + \phi_{n+1}^{\mathrm{K}}\chi_{n+1}^{\mathrm{K}}q_n^{\mathrm{K}}\right) = O \\ &r_{n,t}^{\mathrm{K}} - \mathrm{i}\left(\chi_n^{\mathrm{K}} + \chi_{n+1}^{\mathrm{K}}\right) - 2\left(r_n^{\mathrm{K}}\phi_n^{\mathrm{K}}\chi_n^{\mathrm{K}} + \chi_{n+1}^{\mathrm{K}}\phi_n^{\mathrm{K}}r_n^{\mathrm{K}}\right) = O \\ &\phi_n^{\mathrm{K}} - \phi_{n+1}^{\mathrm{K}} + \mathrm{i}q_n^{\mathrm{K}} = O \\ &\chi_n^{\mathrm{K}} - \chi_{n+1}^{\mathrm{K}} - \mathrm{i}r_n^{\mathrm{K}} = O. \end{aligned} \tag{5.2}$$

The system (5.2) gives an integrable semi-discretization of the massive Thirring model of the Kaup–Newell type (see (A.4) in [29]). As we expect from the property of the semi-discrete Kaup–Newell system (3.1), (5.2) admits either the reduction $r_n^{\mathrm{K}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{K}*}$, $\chi_n^{\mathrm{K}} = \mathrm{i}\sigma\phi_{n-\frac{1}{2}}^{\mathrm{K}*}$ or the reduction $r_n^{\mathrm{K}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{K}\dagger}$, $\chi_n^{\mathrm{K}} = \mathrm{i}\sigma\phi_{n-\frac{1}{2}}^{\mathrm{K}\dagger}$.

We can eliminate q_n^{K} and r_n^{K} to obtain an integrable semi-discretization of the Mikhailov model (see (4.21) in [37] or (3.47) in [27]),

$$\begin{aligned} &\phi_{n,t}^{\mathrm{K}} - \phi_{n+1,t}^{\mathrm{K}} + \phi_n^{\mathrm{K}} + \phi_{n+1}^{\mathrm{K}} + 2\left(\phi_n^{\mathrm{K}}\chi_{n+1}^{\mathrm{K}}\phi_n^{\mathrm{K}} - \phi_{n+1}^{\mathrm{K}}\chi_{n+1}^{\mathrm{K}}\phi_{n+1}^{\mathrm{K}}\right) = O \\ &\chi_{n,t}^{\mathrm{K}} - \chi_{n+1,t}^{\mathrm{K}} + \chi_n^{\mathrm{K}} + \chi_{n+1}^{\mathrm{K}} - 2\left(\chi_n^{\mathrm{K}}\phi_n^{\mathrm{K}}\chi_n^{\mathrm{K}} - \chi_{n+1}^{\mathrm{K}}\phi_n^{\mathrm{K}}\chi_{n+1}^{\mathrm{K}}\right) = O. \end{aligned} \tag{5.3}$$

Through the reduction $\phi_n^{\mathrm{K}} = (-1)^n(\omega_n + I/4)$, $\chi_n^{\mathrm{K}} = (-1)^n I$, (5.3) collapses into the simplest version of the dressing chain [57],

$$\omega_{n+1,t} + \omega_{n,t} + 2\omega_{n+1}^2 - 2\omega_n^2 = O.$$

5.2. Semi-discrete Chen–Lee–Liu type

For the semi-discrete Chen–Lee–Liu system in section 3.2, we replace the M_n-matrix (3.12*b*) with the following one:

$$M_n^{\mathrm{C}} = \frac{1}{z-\frac{1}{z}} \begin{bmatrix} \frac{1}{z} I_1 & 2\mathrm{i} f(z)\phi_n^{\mathrm{C}} \\ \frac{2\mathrm{i}}{f(z)}\left(-1+\frac{1}{z^2}\right)\chi_n^{\mathrm{C}} & -\frac{1}{z}I_2 + 2\left(z-\frac{1}{z}\right)\chi_n^{\mathrm{C}}\phi_n^{\mathrm{C}} \end{bmatrix}. \tag{5.4}$$

Plunging the Lax pair (3.12*a*) and (5.4) into the zero-curvature condition (2.9), we obtain

$$\begin{aligned} &q_{n,t}^{\mathrm{C}} + 2\mathrm{i}\phi_n^{\mathrm{C}} + 2q_n^{\mathrm{C}}\chi_n^{\mathrm{C}}\phi_n^{\mathrm{C}} = O \\ &r_{n,t}^{\mathrm{C}} - 2\mathrm{i}\chi_{n+1}^{\mathrm{C}} - 2\chi_{n+1}^{\mathrm{C}}\phi_{n+1}^{\mathrm{C}}r_n^{\mathrm{C}} = O \\ &\phi_n^{\mathrm{C}} - \phi_{n+1}^{\mathrm{C}} + \mathrm{i}q_n^{\mathrm{C}} + q_n^{\mathrm{C}}r_n^{\mathrm{C}}\phi_n^{\mathrm{C}} = O \\ &\chi_n^{\mathrm{C}} - \chi_{n+1}^{\mathrm{C}} - \mathrm{i}r_n^{\mathrm{C}} - \chi_{n+1}^{\mathrm{C}}q_n^{\mathrm{C}}r_n^{\mathrm{C}} = O. \end{aligned} \tag{5.5}$$

The system (5.5) gives an integrable semi-discretization of the massive Thirring model (see (A.3) in [29]). As we expect from the property of the semi-discrete Chen–Lee–Liu system (3.9), (5.5) admits neither the reduction $r_n^{\mathrm{C}} = \mathrm{i}\sigma q_{n-k}^{\mathrm{C}*}$, $\chi_n^{\mathrm{C}} = \mathrm{i}\sigma \phi_{n-k}^{\mathrm{C}*}$ nor the reduction $r_n^{\mathrm{C}} = \mathrm{i}\sigma q_{n-k}^{\mathrm{C}\dagger}$, $\chi_n^{\mathrm{C}} = \mathrm{i}\sigma \phi_{n-k}^{\mathrm{C}\dagger}$. In section 5.4, we obtain an alternative semi-discretization of the massive Thirring model without this drawback.

5.3. Semi-discrete Gerdjikov–Ivanov type

For the semi-discrete Gerdjikov–Ivanov system in section 3.3, we replace the M_n-matrix (3.15*b*) with the following one:

$$M_n^{\mathrm{G}} = \frac{1}{z - \frac{1}{z}} \begin{bmatrix} \frac{1}{z} I_1 & 2\mathrm{i} f(z) \phi_n^{\mathrm{G}} \\ \frac{2\mathrm{i}}{f(z)} \left(-1 + \frac{1}{z^2}\right) \left(I_2 - r_n^{\mathrm{G}} q_{n-1}^{\mathrm{G}}\right) \chi_n^{\mathrm{G}} & -z I_2 + 2\mathrm{i}\left(z - \frac{1}{z}\right) r_n^{\mathrm{G}} \phi_n^{\mathrm{G}} \end{bmatrix}. \tag{5.6}$$

Putting the Lax pair (3.15*a*) and (5.6) into the zero-curvature condition (2.9), we obtain

$$\begin{aligned} & q_{n,t}^{\mathrm{G}} + \mathrm{i}\left(\phi_n^{\mathrm{G}} + \phi_{n+1}^{\mathrm{G}}\right) + \mathrm{i}\left(q_n^{\mathrm{G}} r_n^{\mathrm{G}} \phi_n^{\mathrm{G}} - \phi_{n+1}^{\mathrm{G}} r_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}}\right) = O \\ & r_{n,t}^{\mathrm{G}} - \mathrm{i}\left(\chi_n^{\mathrm{G}} + \chi_{n+1}^{\mathrm{G}}\right) + \mathrm{i}\left(r_n^{\mathrm{G}} q_{n-1}^{\mathrm{G}} \chi_n^{\mathrm{G}} - \chi_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}} r_n^{\mathrm{G}}\right) = O \\ & \phi_n^{\mathrm{G}} - \phi_{n+1}^{\mathrm{G}} + \mathrm{i} q_n^{\mathrm{G}} + \left(q_n^{\mathrm{G}} r_n^{\mathrm{G}} \phi_n^{\mathrm{G}} + \phi_{n+1}^{\mathrm{G}} r_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}}\right) = O \\ & \chi_n^{\mathrm{G}} - \chi_{n+1}^{\mathrm{G}} - \mathrm{i} r_n^{\mathrm{G}} - \left(r_n^{\mathrm{G}} q_{n-1}^{\mathrm{G}} \chi_n^{\mathrm{G}} + \chi_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}} r_n^{\mathrm{G}}\right) = O. \end{aligned} \tag{5.7}$$

The system (5.7) gives an integrable semi-discretization of the massive Thirring model of the Gerdjikov–Ivanov type (see (A.4) in [29] with interchanging t and x). As we expect from the property of the semi-discrete Gerdjikov–Ivanov system (3.13), (5.7) admits the reduction $r_n^{\mathrm{G}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{G}*}$, $\chi_n^{\mathrm{G}} = \mathrm{i}\sigma \phi_{n-\frac{1}{2}}^{\mathrm{G}*}$ and not the reduction $r_n^{\mathrm{G}} = \mathrm{i}\sigma q_{n-\frac{1}{2}}^{\mathrm{G}\dagger}$, $\chi_n^{\mathrm{G}} = \mathrm{i}\sigma \phi_{n-\frac{1}{2}}^{\mathrm{G}\dagger}$.

If we eliminate ϕ_n^{G} and χ_n^{G}, we obtain another semi-discretization of the Mikhailov model,

$$\begin{aligned} & q_{n+1,t}^{\mathrm{G}}\left(I_2 - r_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}}\right) - \left(I_1 + q_{n+1}^{\mathrm{G}} r_{n+1}^{\mathrm{G}}\right) q_{n,t}^{\mathrm{G}} - q_n^{\mathrm{G}} - q_{n+1}^{\mathrm{G}} = O \\ & r_{n+1,t}^{\mathrm{G}}\left(I_1 + q_n^{\mathrm{G}} r_n^{\mathrm{G}}\right) - \left(I_2 - r_{n+1}^{\mathrm{G}} q_n^{\mathrm{G}}\right) r_{n,t}^{\mathrm{G}} - r_n^{\mathrm{G}} - r_{n+1}^{\mathrm{G}} = O. \end{aligned} \tag{5.8}$$

Comparing (5.8) with (5.3), we note the interchange of space and time. Through the reduction $q_n^{\mathrm{G}} = (-1)^n(\xi_n - I)$, $r_n^{\mathrm{G}} = (-1)^n I$ for (5.8), we obtain the following equation:

$$(\xi_{n+1}\xi_n)_t = \xi_{n+1} - \xi_n.$$

5.4. Alternative semi-discrete Chen–Lee–Liu type

We assume that dependent variables are scalar and consider the transformation (3.17) together with

$$\phi_n = \phi_n^{\mathrm{G}} \prod_{j=-\infty}^{n} \left(\frac{1 - q_{j-1}^{\mathrm{G}} r_j^{\mathrm{G}}}{1 + q_{j-1}^{\mathrm{G}} r_{j-1}^{\mathrm{G}}}\right)^{-2\gamma} \qquad \chi_n = \chi_n^{\mathrm{G}} \prod_{j=-\infty}^{n} \left(\frac{1 - q_{j-2}^{\mathrm{G}} r_{j-1}^{\mathrm{G}}}{1 + q_{j-1}^{\mathrm{G}} r_{j-1}^{\mathrm{G}}}\right)^{2\gamma}. \tag{5.9}$$

Substituting the inverse transformation into (5.7), we obtain an integrable semi-discretization of a generalized massive Thirring model. When $\gamma = -1/2$, the obtained system coincides with (5.2). Setting $\gamma = -1/4$, we obtain an integrable semi-discretization of the massive Thirring model which is different from (5.5). We write the equations for q_n and ϕ_n:

$$\begin{aligned} q_{n,t} &+ \mathrm{i}\left[\sqrt{1 + (q_n r_n)^2} + q_n r_n\right]\phi_n + \mathrm{i}\left[\sqrt{1 + (q_n r_{n+1})^2} - q_n r_{n+1}\right]\phi_{n+1} + 2\left[\sqrt{1 + (q_n r_n)^2}\right. \\ &\left. + q_n r_n\right]\phi_n \chi_{n+1} q_n + 2\left[\sqrt{1 + (q_n r_{n+1})^2} - q_n r_{n+1}\right]\phi_{n+1}\chi_{n+1} q_n = 0 \end{aligned} \tag{5.10a}$$

$$\left[\sqrt{1 + (q_n r_n)^2} + q_n r_n\right]\phi_n - \left[\sqrt{1 + (q_n r_{n+1})^2} - q_n r_{n+1}\right]\phi_{n+1} + \mathrm{i} q_n = 0. \tag{5.10b}$$

Here we have changed the scaling of $q_n r_n$ to eliminate fractions. Unlike (5.5), the alternative semi-discrete massive Thirring model ((5.10) and the equations for r_n and χ_n) admits the reduction $r_n = \mathrm{i}\sigma q^*_{n-\frac{1}{2}}$, $\chi_n = \mathrm{i}\sigma \phi^*_{n-\frac{1}{2}}$.

6. Concluding remarks

In this paper, we have investigated integrable discretizations of the DNLS systems from the point of view of a Lax-pair formulation. We have found that lattice versions of the Kaup–Newell system, the Chen–Lee–Liu system and the Gerdjikov–Ivanov system are embedded in a generalization of the Ablowitz–Ladik system. With appropriate gauge transformations, the eigenvalue problems for the lattice systems coincide with the continuous counterparts in the continuum limit. Using a discrete analogue of the phase transformation (1.6), we obtain a lattice version of the generalized DNLS system by Kund. All the discrete DNLS systems but the semi-discrete Chen–Lee–Liu system (3.9) admit the reduction of complex conjugation between two dependent variables. This property is indispensable for applications such as difference schemes for numerical computation or modelling of nonlinear lattice vibrations. We stress that the discrete DNLS systems have relations to a variety of discrete integrable systems. Through changes of variables and reductions, we obtain integrable discretizations of the mixed NLS, matrix NLS, matrix KdV, matrix mKdV, coupled Sasa–Satsuma, Burgers equations, etc. From the eigenvalue problems for the discrete DNLS systems, we derive integrable discretizations of the massive Thirring-type models.

Besides the Ablowitz–Ladik system, there are other integrable discretizations of the NLS hierarchy. We conjecture that other discrete DNLS systems are embedded in the discrete NLS systems. In fact, we obtain another semi-discretization of the Gerdjikov–Ivanov system in connection with the semi-discrete NLS system studied in [58] (see also [33, 43]). However, the obtained system does not admit the reduction of complex conjugation. Thus it is less attractive than (3.13). The example implies that the Ablowitz–Ladik formulation is an optimal starting point for the study of discrete DNLS systems.

The NLS and DNLS systems for scalar variables possess tri-Hamiltonian structure [59, 60]. The three sets of Poisson brackets are, respectively, the $\delta'(x-y)$ type for the Kaup–Newell system, the $\delta(x-y)$ type for the Chen–Lee–Liu system and the $\delta(x-y)$ type for the NLS system. We should note that all these systems are connected via transformations of the dependent variables. We infer that the lattice versions of these systems studied in this paper possess tri-Hamiltonian structure. The discrete counterpart of the first Hamiltonian structure is (3.6) for the semi-discrete Kaup–Newell system (3.1). The Ablowitz–Ladik system (2.12) has a set of ultra-local Poisson brackets [61–63] which is the discrete analogue of the third Hamiltonian structure. Since the lattice systems are each connected with the others through changes of variables (see section 3), they are at least bi-Hamiltonian. We have not found the discrete counterpart of the second Hamiltonian structure for either the semi-discrete Chen–Lee–Liu system (3.9) or the alternative system (3.21). It remains an unsolved problem to prove explicitly that the Ablowitz–Ladik system and the semi-discrete DNLS systems are tri-Hamiltonian.

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Appendix A. Semi-discrete sine-Gordon equation as a negative flow of the Ablowitz–Ladik hierarchy

In this appendix, we prove that a semi-discrete sine-Gordon equation and its generalization are integrable via the ISM based on the Ablowitz–Ladik eigenvalue problem. We introduce the following form of the Lax pair:

$$L_n = \begin{bmatrix} z & u_n \\ v_n & \frac{1}{z} \end{bmatrix} \qquad M_n = \frac{1}{z - \frac{1}{z}} \begin{bmatrix} \frac{1}{z}a_n & b_n \\ c_n & -za_n \end{bmatrix}. \tag{A.1}$$

Here z is the spectral parameter. u_n, v_n, a_n, b_n, c_n are scalar variables. The M_n-matrix is simpler than that by Ablowitz and Ladik [18], while the L_n-matrix is the same. Substituting (A.1) into the zero-curvature condition (2.9), we obtain the following six equations:

$$a_n - a_{n+1} + u_n c_n - v_n b_{n+1} = 0 \tag{A.2a}$$

$$u_n(c_n - c_{n+1}) + v_n(b_n - b_{n+1}) = 0 \tag{A.2b}$$

$$b_n - b_{n+1} - u_n(a_n + a_{n+1}) = 0 \tag{A.2c}$$

$$c_n - c_{n+1} + v_n(a_n + a_{n+1}) = 0 \tag{A.2d}$$

$$u_{n,t} + b_n - u_n a_n = 0 \tag{A.2e}$$

$$v_{n,t} - c_n - v_n a_n = 0. \tag{A.2f}$$

Thanks to (A.2*c*) and (A.2*d*), we can write u_n and v_n in terms of a_n, b_n, c_n:

$$u_n = \frac{b_n - b_{n+1}}{a_n + a_{n+1}} \qquad v_n = -\frac{c_n - c_{n+1}}{a_n + a_{n+1}}. \tag{A.3}$$

Then (A.2*b*) is automatically satisfied. Putting (A.3) into (A.2*a*), we obtain

$$a_n^2 + b_n c_n = \eta^2 \tag{A.4}$$

where η is a constant.

If we set

$$a_n = \eta \cos\theta_n \qquad b_n = c_n = \eta \sin\theta_n$$

u_n and v_n are expressed as

$$u_n = -v_n = \frac{\sin\theta_n - \sin\theta_{n+1}}{\cos\theta_n + \cos\theta_{n+1}} = \tan\left(\frac{\theta_n - \theta_{n+1}}{2}\right).$$

Then (A.2*e*) and (A.2*f*) are combined into a semi-discrete sine-Gordon equation [64, 65],

$$\theta_{n+1,t} - \theta_{n,t} = \eta(\sin\theta_n + \sin\theta_{n+1}). \tag{A.5}$$

By defining α_n by $\theta_n = (\alpha_n + \alpha_{n+1})/2$, we rewrite (A.5) as [20]

$$\alpha_{n+1,t} - \alpha_{n,t} = 2\eta \sin\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right) \tag{A.6}$$

under appropriate boundary conditions.

More generally, b_n and c_n are functionally independent. Substituting $a_n = \sqrt{\eta^2 - b_n c_n}$ into (A.3), (A.2*e*) and (A.2*f*), we obtain a semi-discretization of the reduced equation for the O(4) nonlinear σ-model [40],

$$\left(\frac{b_n - b_{n+1}}{\sqrt{\eta^2 - b_n c_n} + \sqrt{\eta^2 - b_{n+1}c_{n+1}}}\right)_t + \frac{b_n\sqrt{\eta^2 - b_{n+1}c_{n+1}} + b_{n+1}\sqrt{\eta^2 - b_n c_n}}{\sqrt{\eta^2 - b_n c_n} + \sqrt{\eta^2 - b_{n+1}c_{n+1}}} = 0 \tag{A.7a}$$

$$\left(\frac{c_n - c_{n+1}}{\sqrt{\eta^2 - b_n c_n} + \sqrt{\eta^2 - b_{n+1}c_{n+1}}}\right)_t + \frac{c_n\sqrt{\eta^2 - b_{n+1}c_{n+1}} + c_{n+1}\sqrt{\eta^2 - b_n c_n}}{\sqrt{\eta^2 - b_n c_n} + \sqrt{\eta^2 - b_{n+1}c_{n+1}}} = 0. \tag{A.7b}$$

If we set $\eta = 0$ and $b_n = -c_n = \mathrm{e}^{\phi_n}$, (A.7) is simplified to a semi-discrete Liouville equation [66],

$$\phi_{n+1,t} - \phi_{n,t} = \mathrm{e}^{\phi_n} + \mathrm{e}^{\phi_{n+1}}.$$

We have shown that the semi-discrete sine-Gordon equation (A.5) and its generalization (A.7) are associated with the Ablowitz–Ladik eigenvalue problem. Thus they are integrated by the ISM straightforwardly. The semi-discrete sine-Gordon equation (A.5) is associated with another ISM-solvable eigenvalue problem [64, 65]. It is now clear that the difference between the two eigenvalue problems is not essential (see also [21]). We obtain a lot of knowledge on the semi-discrete sine-Gordon equation and its generalization from the study of the Ablowitz–Ladik hierarchy.

In section 5, we investigate integrable semi-discretizations of the massive Thirring-type models. The M_n-matrix in (A.1) with the constraint (A.4) gives helpful information for that.

Appendix B. Time discretizations of semi-discrete sine-Gordon equation

In this appendix, we discuss time discretizations of the semi-discrete sine-Gordon equation and its generalization obtained in appendix A. We consider a time discretization of (2.8)

$$\Psi_{n+1} = L_n \Psi_n \qquad \tilde{\Psi}_n = V_n \Psi_n \tag{B.1}$$

where the tilde denotes the step-up shift ($l \rightarrow l+1$) in the discrete time $l \in \mathbb{Z}$. The compatibility of (B.1) leads to a full-discrete version of the zero-curvature condition [63, 67]:

$$\tilde{L}_n V_n = V_{n+1} L_n. \tag{B.2}$$

As a discrete-time analogue of (A.1), we introduce the following form of the Lax pair:

$$L_n = \begin{bmatrix} z & u_n \\ v_n & \frac{1}{z} \end{bmatrix} \qquad V_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{h}{z - \frac{1}{z}} \begin{bmatrix} \frac{1}{z} a_n & b_n \\ c_n & -z a_n \end{bmatrix}. \tag{B.3}$$

Here h denotes the difference interval of time. Substituting (B.3) into the zero-curvature condition (B.2), we obtain the following six equations:

$$a_n - a_{n+1} + \tilde{u}_n c_n - v_n b_{n+1} = 0 \tag{B.4a}$$

$$\tilde{u}_n c_n - u_n c_{n+1} + \tilde{v}_n b_n - v_n b_{n+1} = 0 \tag{B.4b}$$

$$b_n - b_{n+1} - \tilde{u}_n a_n - u_n a_{n+1} = 0 \tag{B.4c}$$

$$c_n - c_{n+1} + \tilde{v}_n a_n + v_n a_{n+1} = 0 \tag{B.4d}$$

$$\tilde{u}_n - u_n + h(b_n - \tilde{u}_n a_n) = 0 \tag{B.4e}$$

$$\tilde{v}_n - v_n - h(c_n + \tilde{v}_n a_n) = 0. \tag{B.4f}$$

Thanks to (B.4c) and (B.4e), we can write u_n and $\tilde{u}_n$ in terms of a_n and b_n:

$$u_n = \frac{b_n - b_{n+1} + h a_n b_{n+1}}{a_n + a_{n+1} - h a_n a_{n+1}} \qquad \tilde{u}_n = \frac{b_n - b_{n+1} - h a_{n+1} b_n}{a_n + a_{n+1} - h a_n a_{n+1}}. \tag{B.5a}$$

Similarly, using (B.4d) and (B.4f), we express v_n and $\tilde{v}_n$ in terms of a_n and c_n:

$$v_n = -\frac{c_n - c_{n+1} + h a_n c_{n+1}}{a_n + a_{n+1} - h a_n a_{n+1}} \qquad \tilde{v}_n = -\frac{c_n - c_{n+1} - h a_{n+1} c_n}{a_n + a_{n+1} - h a_n a_{n+1}}. \tag{B.5b}$$

It is easily seen that (B.4b) is automatically satisfied. Putting (B.5) into (B.4a), we obtain $a_n^2 + b_n c_n = \eta^2 (1 - h a_n)$ or, equivalently,

$$\left(a_n + \frac{h\eta^2}{2} \right)^2 + b_n c_n = \eta^2 \left[1 + \left(\frac{h\eta}{2} \right)^2 \right]. \tag{B.6}$$

Here η is a constant.

Relations (B.5) lead to the following system,

$$\frac{\tilde{b}_n - \tilde{b}_{n+1} + h\tilde{a}_n\tilde{b}_{n+1}}{\tilde{a}_n + \tilde{a}_{n+1} - h\tilde{a}_n\tilde{a}_{n+1}} - \frac{b_n - b_{n+1} - ha_{n+1}b_n}{a_n + a_{n+1} - ha_na_{n+1}} = 0 \tag{B.7a}$$

$$\frac{\tilde{c}_n - \tilde{c}_{n+1} + h\tilde{a}_n\tilde{c}_{n+1}}{\tilde{a}_n + \tilde{a}_{n+1} - h\tilde{a}_n\tilde{a}_{n+1}} - \frac{c_n - c_{n+1} - ha_{n+1}c_n}{a_n + a_{n+1} - ha_na_{n+1}} = 0 \tag{B.7b}$$

where a_n, b_n, c_n are related through (B.6). Substituting

$$a_n = -\frac{h\eta^2}{2} + \eta\sqrt{1+\left(\frac{h\eta}{2}\right)^2}\cos\theta_n \qquad b_n = c_n = \eta\sqrt{1+\left(\frac{h\eta}{2}\right)^2}\sin\theta_n$$

into (B.7), we obtain an integrable time discretization of the semi-discrete sine-Gordon equation (A.5). More generally, putting

$$a_n = -\frac{h\eta^2}{2} + \sqrt{\eta^2 + \left(\frac{h\eta^2}{2}\right)^2 - b_nc_n}$$

into (B.7), we obtain an integrable time discretization of the semi-discrete reduced nonlinear σ-model (A.7).

We shall extract a celebrated full-discrete sine-Gordon equation from (B.4). The equation is interpreted as an integrable time discretization of (A.6). We introduce the following parametrization:

$$a_n = -\frac{h\eta^2}{2} + \eta\sqrt{1+\left(\frac{h\eta}{2}\right)^2}\cos\left(\frac{\tilde{\alpha}_n + \alpha_{n+1}}{2}\right) \tag{B.8a}$$

$$b_n = c_n = \eta\sqrt{1+\left(\frac{h\eta}{2}\right)^2}\sin\left(\frac{\tilde{\alpha}_n + \alpha_{n+1}}{2}\right) \tag{B.8b}$$

$$u_n = -v_n = \tan\left(\frac{\alpha_n - \alpha_{n+2}}{4}\right). \tag{B.8c}$$

It is sufficient to consider (B.4*a*), (B.4*c*), (B.4*e*) among (B.4). For brevity, we employ the following abbreviations:

$$x_n = \frac{\tilde{\alpha}_n + \alpha_{n+1}}{4} \qquad y_n = \frac{\alpha_n + \tilde{\alpha}_{n+1}}{4} \qquad k = 1 + \frac{(h\eta)^2}{2} + h\eta\sqrt{1+\left(\frac{h\eta}{2}\right)^2}.$$

Under the parametrization (B.8), we rewrite (B.4*a*), (B.4*c*), (B.4*e*) respectively as

$$\frac{\cos x_n \cos y_{n+1}}{\cos(x_n - y_{n+1})} = \frac{\cos y_n \cos x_{n+1}}{\cos(y_n - x_{n+1})} \tag{B.9a}$$

$$\frac{k\sin x_n \cos y_{n+1} + \cos x_n \sin y_{n+1}}{\cos(x_n - y_{n+1})} = \frac{\sin y_n \cos x_{n+1} + k\cos y_n \sin x_{n+1}}{\cos(y_n - x_{n+1})} \tag{B.9b}$$

$$\frac{k\sin x_n \cos y_{n+1} - k^{-1}\cos x_n \sin y_{n+1}}{\cos(x_n - y_{n+1})} = \frac{\sin y_n \cos x_{n+1} - \cos y_n \sin x_{n+1}}{\cos(y_n - x_{n+1})}. \tag{B.9c}$$

With the help of (B.9*a*) we combine (B.9*b*) and (B.9*c*) into one equation:

$$k\tan x_n = \tan y_n. \tag{B.10}$$

Conversely, all of the relations (B.9) hold if x_n and y_n satisfy (B.10). Equation (B.10) is the celebrated full-discrete sine-Gordon equation [68]. It is easily cast into the standard form [19],

$$\sin\left(\frac{\alpha_n + \tilde{\alpha}_{n+1} - \tilde{\alpha}_n - \alpha_{n+1}}{4}\right) = \frac{\frac{h\eta}{2}}{\sqrt{1 + \left(\frac{h\eta}{2}\right)^2}} \sin\left(\frac{\alpha_n + \tilde{\alpha}_{n+1} + \tilde{\alpha}_n + \alpha_{n+1}}{4}\right).$$

We have obtained integrable full-discretizations of the sine-Gordon equation and the reduced nonlinear σ-model. The obtained equations are integrable via the ISM based on the Ablowitz–Ladik eigenvalue problem.

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